The branching problem in generalized power solutions to differential equations

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Abstract

Generalized power asymptotic expansions of solutions to differential equations that depend on parameters are investigated. The changing nature of these expansions as the parameters of the model cross critical values is discussed. An algorithm to identify these critical values and generate the generalized power series for distinct families of solutions is presented, and as an application the singular behavior of a cosmological model with a nonlinear dissipative fluid is obtained. This algorithm has been implemented in the computer algebra system Maple.

Key words: Generalized power series, Nonlinear ordinary differential equations, Symbolic computation, Cosmological models
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1 Introduction

Quite frequently physical models depend on parameters and the predictions of these models, when confronted with observations, yield constraints that fix the parameters or disqualify the model. Hence, for models described in terms of a set of differential equations, it becomes of prime importance to investigate the dependence of solutions on the parameters. Now, when exact solutions to these equations are not available, or even when they are available but they have a complex implicit or parametric form, it may become useful to obtain asymptotic expansions of these solutions. And when solutions do not admit power series expansions with integer or rational exponents (Pusieux series), we are led to consider series with more general terms like "generalized" powers with real exponents [1] or exp-log terms [2]. Though the algebraic issues of generalized power series have already been investigated (cf. [3]), for applications the main problem lies in the dependence of the exponents on the

parameters, as the ordering of terms and even the nature of the asymptotic expansion may change as these parameters reach some critical values.

In this paper, we will discuss the problems involved in the determination of the critical values of the parameters and an algorithm to calculate the coefficients and exponents of the generalized power series in the regions of the parameter space where real solutions admit such an expansion.

Several critical steps in the search for solutions in the form of generalized power expansions involve calculations with large number of terms and this number grows very fast with the size of the ODE, making hand calculation inconvenient. On the other hand, as many of the steps in these calculations have a systematic nature, use of computer algebra systems appears ideally suited. For this purpose, we have developed a set of routines in Maple (for a brief review of a previous implementation, see Ref. [4]).

The plan of this paper is as follows. In section 2, we present the basics of generalized power expansions of solutions to a class of nonlinear ODES. In section 3, we review the problem of branching of these solutions as parameters cross critical values. The algorithm used to find these solutions is presented in section 4 and it is applied to an example equation arising from a cosmological model in section 5. Finally, the conclusions are stated in section 6.

2   The generalized power expansion

We will consider nonlinear ordinary differential equations of the form

$$D[y(t)] = \sum_{i=1}^{N} A_i \, y^{B_i^0} \left(\frac{dy}{dt}\right)^{B_i^1} \cdots \left(\frac{d^r y}{dt^r}\right)^{B_i^r} = 0 \tag{1}$$

where the coefficients A_i and the exponent B_i^0 may depend on parameters $p_1, \ldots, p_Q$. In the cosmological setting, y is frequently a monotonic function of either the scale factor or the Hubble rate in an expanding universe (cf. [5] [6] [7] [8]). Hence $y(t) > 0$ and equation (1), if not algebraic, is still well defined.

In case that the general solution to (1) is not available, we are usually interested in obtaining some information about it in the form of an asymptotic expansion for the limits $t \to 0^+$ and $t \to \infty$, of the form

$$y(t) \sim \sum_{j=1}^{\infty} c_j t^{n_j} \tag{2}$$

In homogeneous cosmological models, t is the universal time and these limits frequently correspond to the behavior of the solution near the initial singularity (the "Big Bang") or at large time. Here, c_j and n_j are real, in principle functions of $p_1, \ldots, p_Q$, $c_j \neq 0$ and the exponents n_j form an ordered set: $n_1 < n_2 < \cdots$ for $t \to 0^+$ and $n_1 > n_2 > \cdots$ for $t \to \infty$. So $c_1 t^{n_1}$ is the leading term, $n_1 \neq 0$, $t^{n_{j+1}}/t^{n_j} \to 0$ in either limit and the set $\{t^{n_j}\}$ constitutes an asymptotic scale. Inserting the series (2) into equation (1), and performing the necessary asymptotic expansions, we get another generalized power series

$$D \left[\sum_{j=1}^{\infty} c_j t^{n_j} \right] \sim \sum_{k=1}^{\infty} C_k t^{e_k} \tag{3}$$

where C_k and e_k are also real functions of $p_1, \ldots, p_Q$, and the exponents e_k form an ordered set: $e_1 < e_2 < \cdots$ for $t \to 0^+$ and $e_1 > e_2 > \cdots$ for $t \to \infty$. For the equation (1) to admit a solution with expansion (2), each of the C_k must vanish and this set of equations fixes in principle the c_j, n_j in (2) up to $r - 1$ of them that remain free and arise from the integration constants of the general solution of (1) (the arbitrary constant corresponding to the time translational symmetry is fixed to 0). Each set of solutions $\{c_j, n_j\}$ yields a series representation of a family of solutions to (1). Further, the constraints that these c_j, n_j are real and the n_j are ordered may delimit regions in parameter space where generalized power law solutions are feasible. If these regions do not contain the values of the parameters that make physical sense, such solutions have to be discarded even when they are mathematically correct.

3 Case branching by parameter variation

For simplicity, let us consider that equation (1) depends on a single parameter p, so that $n_j = n_j(p)$ and $c_j = c_j(p)$. As the exponents $\{n_j\}$ have to be ordered, a critical value p_0 of the parameter arises when a pair of consecutive exponents become equal, $n_{j+1}(p_0) = n_j(p_0)$ say. Moving the parameter p across p_0 may cause changes in the nature of the solution, hence in its series expansion, e.g., the asymptotic scale involved, and these effects show up on the behavior of the function $\nu_j(p) \equiv n_{j+1}(p) - n_j(p)$ in a neighbourhood of p_0. Let us consider first that $\nu_j(p)$ is analytic at p_0 so that, excluding very special cases, $\nu_j = d_j(p - p_0) + \cdots$ holds with some constant $d_j \neq 0$. Hence, the terms labeled j and $j + 1$ on one side of p_0 switch order on the other side. This effect is of particular importance, as regards to the analysis of the families of solutions, when the leading behavior changes. For the series at p_0, let us make the expansion of this pair of terms

$$c_j t^{n_j} + c_{j+1} t^{n_{j+1}} = t^{n_j} \left[c_j + c_{j+1} + c_{j+1} d_j (p - p_0) \ln t + \cdots \right] \tag{4}$$

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If $\lim c_{j+1}(p - p_0) \neq 0$ for $p \to p_0$, as when c_{j+1} is an arbitrary constant, the series picks a logarithmic term, and this case has to be dealt with separately. Otherwise, both terms merge into one and a relabeling of terms occur.

On the other hand, when p_0 is a branching point of ν_j as in $\nu_j = d_j(p_0 - p)^{s/r} + \cdots$, with s odd and r even, an expansion of the real solution as a generalized power series exists only on one side and the nature of the solution changes when p_0 is crossed. As an example, let us consider the equation

$$\ddot{y} + y\dot{y} + \beta y^3 = 0 \tag{5}$$

with parameter β, that arises in several cosmological models (e.g., [9][10][11][13]), as well as in the analysis of the Painlevé equations [12], and in form invariant equations [15] [16]. It may be shown that the asymptotic expansion of the general solution to (5) is

$$y(t) \sim \frac{\alpha}{t} \sum_{n=0}^{\infty} c_n \gamma^n t^{nr} \tag{6}$$

where $\alpha_\pm = [1 \pm (1 - 8\beta)^{1/2}]/(2\beta)$, $r = 4 - \alpha$ is the Kowalevski exponent [17], $r > 0$ for $t \to 0^+$ and $r < 0$ for $t \to \infty$, $c_0 = 1$, $c_n = c_n(\beta)$ and γ is an arbitrary integration constant. Here, the critical value is $\beta_0 = 1/8$ and $\nu_j = r_\pm = \mp 8[2(\beta_0 - \beta)]^{1/2} + \cdots$. Hence, real solutions admit expansion (6) only for $\beta < 1/8$. For $\beta = 1/8$, the general solution has a logarithmic expansion and for $\beta > 1/8$, real solutions become oscillatory.

As the coefficients c_j have to be real and nonvanishing, a critical value p_0 of the parameter arises when it is a root of a coefficient $c_j(p_0) = 0$ or it is a branching point as in $c_j(p) = c_{j0} + d_j(p_0 - p)^{s/r} + \cdots$, with c_{j0} and d_j some constants, s odd and r even. The expansion (6) to the solution of equation (5) shows the second effect as its coefficients are proportional to α, hence they are complex for $\beta > 1/8$.

4   The algorithm

We review briefly the algorithm stated in Ref. [4]. The objective of the calculations is to obtain a truncation of (2) to a finite number of terms, say M

$$y_M(t) = \sum_{j=1}^{M} c_j t^{n_j} \tag{7}$$

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The method of calculation is iterative, so that constants c_M, n_M, $M > 1$, when not free, are determined by the constants $c_1, n_1, \ldots, c_{M-1}, n_{M-1}$ that were found in the previous steps.

We start with $M = 1$, by inserting $y_1 = c_1 t^{n_1}$ into (1). After collecting all the terms with the same generalized power, we get a sum of the form

$$D[y_1(t)] = \sum_{l=1}^{R} D_l\, t^{f_l} \tag{8}$$

We note that, in general, the set of the exponents $F = \{f_l\}_{1 \le l \le R}$ is not ordered. The i-th term of (1) contributes to (3) with a term of exponent

$$g_i = \sum_{h=0}^{r} (n_1 - h)\, B_i^h \tag{9}$$

and coefficient

$$E_i = A_i c_1^{\sum_{h=0}^{r} B_i^h} \prod_{h=1}^{r} (n_1 - h + 1)^{\sum_{j=h}^{r} B_i^j} \equiv \nu_i c_1^{\mu_i} \tag{10}$$

As more than one term of (1) may yield the same exponent (balance), we have $R \le N$. The exponents g_i, hence the exponents f_l, depend linearly on n_1, and the pair of terms l and m of (8), to which respectively contribute the terms $i(l)$ and $i'(m)$ of (1), balance for the exponent n_{1e}^{lm}

$$n_{1e}^{lm} = \frac{\sum_{h=0}^{r} h \left[B_{i(l)}^h - B_{i'(m)}^h \right]}{\sum_{h=0}^{r} \left[B_{i(l)}^h - B_{i'(m)}^h \right]} \tag{11}$$

We note that both the exponents g_i and the coefficients E_i may also depend on the parameters $p_1, \ldots, p_Q$ through the exponents B_i^0 and the coefficients A_i. In addition, we note that the freedom of multiplying the equation (1) by integer powers of y and its derivatives (up to missing solutions where they vanish), that change its exponents by $B_i^h \to B_i^h + m^h$ for some integers m^h, implies that the exponents in (9) and (10) are representatives of a class of exponents related by the transformation laws $g_i \to g_i + \sum_{h=0}^{r} (n_1 - h)\, m^h$ and $\mu_i \to \mu_i + \sum_{h=0}^{r} m^h$.

If the equation (1) admits a solution of the form (2), and $y_1(t)$ is its leading term, it holds asymptotically $y(t) \sim y_1(t)$. Hence on the one hand

$$D[y(t)] \sim C_1 t^{e_1} \sim D_{l_1} t^{f_{l_1}} \tag{12}$$

for some index l_1 that we identify by sorting the exponents in the set F. For simplicity, only the behavior for $t \to 0^+$ will be considered in the following. In this case we get $e_1 = f_{l_1} \equiv \min(f_l)_{1 \leq l \leq R}$.

On the other hand we require $c_1 \neq 0$ and $D_{l_1} = 0$. As f_{l_1} is the minimum of the $\{g_i\}_{1 \leq i \leq N}$, and we can choose $\mu_i \neq 0$, at least a pair of terms in (1) must yield this same exponent. Let us say that this minimum occurs for $i \in I$, so that $C_1 = D_{l_1} = \sum_{i \in I} E_i$ and this coefficient is a function of c_1 and n_1.

Then, by solving the set of equations $f_l(n_1) = f_m(n_1)$, $1 \leq l < m \leq R$, for n_1, we obtain the set N_e of equality exponents n_{1e} that make the exponents f_l exchange order. They delimit intervals within which the sorting of F has to be carried out separately. Two cases arise: (a) n_1 lays inside any of these intervals, (b) n_1 is an equality exponent. For each interval and equality exponent F is sorted and the coefficient of the term with the minimum exponent f_{l_1} is identified. In case (b) it is verified whether $D_{l_1}(c_1) = 0$ is satisfied with a nonvanishing root c_1, while in case (a) the equation $D_{l_1} = 0$ could determine n_1 provided that a nonvanishing c_1 is feasible. Each pair of real numbers $(c_1, n_1) \neq (0, 0)$ obtained from this analysis corresponds to the leading term of the expansion of a family of solutions. Thus, subsequent calculations to obtain higher order terms must proceed separately for each pair. A constant not fixed by this procedure corresponds in principle to an integration constant of $y(t)$.

Additional branching occurs due to the dependence on the parameters. As shown in (11), the equality exponents depend on $p_1, \ldots, p_Q$ through the B_i^0. If a pair of equality exponents exist, $n_{1e}^{(1)}$ and $n_{1e}^{(2)}$ say, and they themselves become equal, the equation $n_{1e}^{(1)}(p_1, \ldots, p_Q) = n_{1e}^{(2)}(p_1, \ldots, p_Q)$ defines a hypersurface in the parameter space (as the exponents B_i^0 may depend on a subset of the parameters, only a parameter subspace might need to be considered, and when they depend just on a single parameter, hypersurfaces become its critical values). Besides, another set of hypersurfaces in the parameter space might exist where the equality exponents diverge. Hence, the analysis described before has to be done separately inside each of the regions of the parameter space delimited by these hypersurfaces, and at their boundaries.

For $M \geq 2$, when inserted (7) into (1), any factor $y_M^{(h)B_i^h}$, $h = 0, \ldots, r$, with a nonnegative integer exponent B_i^h needs to be expanded, or else expanded asymptotically up to order M, producing a term of the form $K_1 t^{(n_1 - h)B^h}(1 + \cdots + K_M t^{n_M - n_1})$, with some constants K_j. Then, after such expansions, crossed terms in the products generate additional terms with larger exponents. For example, when $M = 2$, a term generated by all factors from the leading term except one has an exponent $g_i' = g_i + n_2 - n_1 > g_i$. Thus we see that only the leading term of (2) can contribute to the leading term of (3).

Once the solution coefficients $c_1, \ldots, c_{M-1}$ and exponents $n_1, \ldots, n_{M-1}$ (for a family of solutions) are determined, the main tasks at step M are:

(i) Insert y_M in (1) and expand (asymptotically to order M) the powers.

(ii) Collect all the terms with the same power of t.

Thus we arrive at an expression of the form

$$D\left[y_M(t)\right] = \sum_{l=1}^{R_M} D_l\, t^{f_l} \tag{13}$$

where D_l and f_l are real functions of $p_1, \ldots, p_Q$, and $R_M \leq NM^{r+1}$ as some terms in (1) may balance. In general, the sequence of exponents $f_1, f_2, \ldots, f_{R_M}$ is not ordered. As c_M and n_M only appear in C_k for $k \geq M$, the first M terms of (13), once sorted after the order of the exponents, are equal to the first M terms of expansion (3). In particular, the first $M-1$ terms are those already found in step $M-1$ of the iteration. Then, to identify the candidates for the exponent e_M we

(iii) sort the set of exponents $F_M = \{f_l\}_{1 \leq l \leq R_M}$, and pick the M-th exponent f_{l_M}.

If the equation (1) admits a solution of the form (2), and $y_M(t)$ is its M-term truncation, we get $e_M = f_{l_M}$. The sorting operation is the most involved part of the whole calculation because of case branching. As $f_l = f_l(n_M)$, those solution exponents $\{n_{Me}\}$ that make a pair of exponents equal, $f_l(n_{Me}^{lm}) = f_m(n_{Me}^{lm})$ say, and satisfy $n_{M-1} < n_{Me}$, separate intervals where a given sorting holds. For $M > 1$ these equality exponents n_{Me} arise as solutions to equations of the form

$$\sum_{j=1}^{M} \left(\alpha_l^j - \alpha_m^j\right) n_j + \sum_{i=1}^{N} \sum_{h=0}^{r} \left(\beta_l^{ih} - \beta_m^{ih}\right) B_i^h = 0 \tag{14}$$

where the α_l^j are linear functions of the B_i^h (see [4] for a geometric interpretation of this equation). Furthermore, hypersurfaces in the parameter space arise as pairs of these equality solution exponents n_{Me} become equal. In (14) they enter through the B_i^0 as well as through $n_1, \ldots, n_{M-1}$. Hence, step (iii) further divides into:

(iiia) Find the set of the the equality exponents $N_e = \{n_{Me}\}$.

(iiib) Identify the hypersurfaces in the parameter space where the equality exponents become equal or diverge.

(iiic) Sort the $\{f_l\}$ for each distinct case.

The next steps are:

(iv) Find n_M (if possible) and C_M.

(v) Solve $C_M = 0$ for either c_M or n_M (if not determined in step (iv)).

A set of routines to deal with steps (iii) and (iv) have been developed in Maple.

5 Example

We will show in this section the application of the algorithm sketched in section 4 to an equation that depends on several parameters and shows some of the issues discussed in the previous sections.

In order to treat dissipative processes in cosmology which are not close to equilibrium, a nonlinear phenomenological generalization of the Israel-Stewart theory was developed recently [18]. Scenarios in which this kind of processes may have occurred include inflation driven by a viscous stress [18,19], and the reheating era at the end of inflation [13,14]. In a spatially flat Friedmann-Lemaitre-Robertson-Walker universe, Einstein's equations together with state and transport equations of the fluid give the evolution equation for the Hubble rate H [18]

$$\left[1 - \frac{k^2}{v^2} - \left(\frac{2k^2}{3\gamma v^2}\right) \frac{\dot{H}}{H^2}\right] \left\{\dddot{H} + 3H\dot{H} + \left(\frac{1 - 2\gamma}{\gamma}\right) \frac{\dot{H}^2}{H} + \frac{9}{4}\gamma H^3\right\}$$
$$+ \frac{3\gamma v^2}{2\alpha} \left[1 + \left(\frac{\alpha k^2}{\gamma v^2}\right) H^{q-1}\right] H^{2-q} \left(2\dot{H} + 3\gamma H^2\right) - \frac{9}{2}\gamma v^2 H^3 = 0 \qquad (15)$$

where γ, α, v, q and k are parameters describing the thermodynamical properties of the fluid. We consider an ordinary viscous fluid so that $1 \le \gamma \le 2$, $0 < v < 1$, $\alpha > 0$ and $k > 0$. In the following, we will calculate two term truncations of the generalized power expansions of the solutions to equation (15) in the limit $t \to 0^+$, corresponding to the behavior of the solutions near the initial singularity.

	$n_1 \leq -1$	$(7-q)n_1$
$q < 1$	$-1 < n_1 \leq \dfrac{2}{q-3}$	$6n_1 - n_1 q - 1$
	$n_1 > \dfrac{2}{q-3}$	$3n_1 - 3$
$1 < q \leq 3$	$n_1 \leq -1$	$6n_1$
	$n_1 > -1$	$3n_1 - 3$
$q > 3$	$n_1 \leq -1$	$6n_1$
	$-1 < n_1 \leq \dfrac{2}{q-3}$	$3n_1 - 3$
	$n_1 > \dfrac{2}{q-3}$	$6n_1 - n_1 q - 1$

Table 1
Table of leading exponents

5.1 Leading term

We begin by inserting the leading term $H = c_1 t^{n_1}$ into (15), looking for solutions with $n_1 < 0$ and $c_1 > 0$. We get the set of exponents

$$F_1 = \{6\,n_1,\; 4\,n_1 - 2,\; 5\,n_1 - 1,\; 3\,n_1 - 3,\; -n_1\,(q-7),\; 6\,n_1 - n_1\,q - 1\}$$

and the set of equality exponents

$$N_{1e} = \left\{ -1,\; \frac{2}{q-3},\; \frac{1}{q-2},\; \frac{3}{q-4},\; -\frac{1}{q} \right\}$$

These, in turn, are function of q, and we find that there is a critical value $q = 1$ that make them equal. In effect, this is a distinct value as all terms of (1) balance and $H = c_1/t$ is an exact solution with c_1 given by the real roots of the cubic equation

$$\frac{9\gamma}{2}\left[\frac{\gamma v^2}{\alpha} + \left(v^2 - \frac{1}{2} \right)\left(\frac{k^2}{v^2} - 1 \right) \right] c_1{}^3 +$$

$$3\left[\left(\frac{3}{2} - v^2 \right)\frac{k^2}{v^2} - 1 - \frac{\gamma\, v^2}{\alpha} \right] c_1{}^2 + \frac{1}{\gamma}\left(1 - 3\frac{k^2}{v^2} \right) c_1 + \frac{2k^2}{3\gamma^2 v^2} = 0$$

This shows that $q = 1$ delimits different behaviors of the solutions (cf. [20]). Besides, as an equality exponent diverges for $q = 0, 2, 3, 4$, sorting of F_1 must be done separately at these values of q and within the intervals they delimit, namely $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, $(2, 3)$, $(3, 4)$, and $(4, \infty)$. The Table 1 shows the leading exponent for each case.

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Thus, we find that the leading exponent switch at $n_1 = -1$, as well as $n_1 = 2/(q-3)$ for $q < 1$ or $q > 3$, where terms balance. Let us start with these equality exponents. For $n_1 = -1$ and $q < 1$, we have $C_1 = D(q-7)$ (we denote D_l by $D(f_l)$) where

$$D(q-7) = \left(-3\,\frac{c_1{}^{(6-q)}\,\gamma}{\alpha} + \frac{9}{2}\,\frac{c_1{}^{(7-q)}\,\gamma^2}{\alpha}\right)v^2$$

so that $c_1 = 2/(3\gamma)$ is a leading coefficient. For $q > 1$, we have $C_1 = D(-6)$ where

$$D(-6) = \frac{9\gamma}{2}\left(v^2 - \frac{1}{2}\right)\left(\frac{k^2}{v^2} - 1\right)c_1{}^3 +$$

$$3\left[\left(\frac{3}{2} - v^2\right)\frac{k^2}{v^2} - 1\right]c_1{}^2 + \frac{1}{\gamma}\left(1 - 3\frac{k^2}{v^2}\right)c_1 + \frac{2k^2}{3\gamma^2 v^2}$$

so that we get three leading coefficients

$$c_1 = \frac{2k^2}{3\gamma\,(k^2 - v^2)}, \qquad \frac{2}{3\gamma\left(1 \pm \sqrt{2}v\right)} \tag{16}$$

provided that $k > v$ in the first case and $v < 1/\sqrt{2}$ in the third one. For the other equality exponent $n_1 = 2/(q-3)$ and $q < 1$ or $q > 3$, we have $C_1 = D(3(5-q)/(q-3))$ where

$$D\left(3\frac{5-q}{q-3}\right) = 6\,\frac{c_1{}^{(6-q)}\,\gamma\,v^2}{\alpha\,(q-3)} + \frac{8k^2}{3\gamma v^2(q-3)^3}\left(q - 1 - \frac{2}{\gamma}\right)c_1^3$$

so that

$$c_1 = \left[\frac{9\gamma^3\,v^4\,(9 - 6\,q + q^2)}{4k^2\,\alpha\,(\gamma - \gamma\,q + 2)}\right]^{\frac{1}{q-3}} \tag{17}$$

is the leading coefficient of another family of solutions provided that $(9 - 6\,q + q^2)/(\gamma - \gamma\,q + 2) > 0$. The remaining cases in Table 1 yield

$$D((7 - q)n_1) = \frac{9c_1{}^{7-q}\gamma^2 v^2}{2\alpha}$$

$$D(6n_1 - n_1 q - 1) = 3\,\frac{c_1{}^{6-q}\gamma\,v^2 n_1}{\alpha}$$

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$$D(3n_1 - 3) = \frac{2n_1^2}{3\gamma v^2}\left(n_1 + 1 - \frac{n_1}{\gamma}\right)$$

$$D(6n_1) = \frac{9\gamma c_1^6}{2}\left(\frac{1}{2} - v^2\right)\left(1 - \frac{k^2}{v^2}\right)$$

We see that no solution exists in the first two cases, the third case provides a solution with $n_1 = -\gamma/(\gamma - 1)$, if $\gamma \neq 1$, and c_1 arbitrary, while the fourth case shows that $D(6n_1) = 0$ if $k = v$ or $v = 1/\sqrt{2}$. For these values, the calculation of the leading term has to be done again.

5.2   Subleading term

Inserting $H(t) = 2/(3\gamma t) + c_2 t^{n_2}$ into (15) and expanding the terms with noninteger exponents we get the sets of exponents

$$F_2 = \{-6, 4n_2 - 2, q + n_2 - 6, q + 2n_2 - 5, 5n_2 - 1, 3n_2 - 3, 6n_2, n_2 - 5, 2n_2 - 4\}$$

and the set of equality exponents larger than -1, sorted for $q < 1$

$$N_{2e}^< = \left\{-\frac{1}{2} - \frac{q}{2}, -q\right\}$$

Thus we find that $e_2 = q + n_2 - 6$ for $n_2 \leq -q$, while $e_2 = -6$ for $n_2 > -q$ and $q < 1$. In the balancing case $n_2 = -q$, we find $C_2 = D(-6)$ where

$$D(-6) = -\frac{4v^2}{3\gamma^2\,\alpha}\left[\alpha + \gamma(q - 2)\left(\frac{3\gamma}{2}\right)^q c_2\right]$$

so that a solution exists with the subleading coefficient

$$c_2 = \frac{\alpha}{\gamma\,(2 - q)}\left(\frac{2}{3\,\gamma}\right)^q \tag{18}$$

For the rest of the cases, we have

$$D(q - 6 + n_2) = \frac{32}{81}\frac{3^q 2^{-q}\gamma^{-4+q}\,(n_2 + 2)\,v^2 c_2}{\alpha}$$

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$$D(-6) = -\frac{32}{81}\frac{v^2}{\gamma^5}$$

and neither of them provides a solution.

Following similar steps, inserting $H(t) = c_1/t + c_2 t^{n_2}$, with c_1 given by (16), into (15) yields $e_2 = n_2 - 5$ for $n_2 \le q - 2$, while $e_2 = q - 7$ for $n_2 > q - 2$. At the equality exponent $n_2 = q - 2$, we obtain the subleading coefficients corresponding to the three cases of (16)

$$c_2 = -\frac{8k^{6-2q}\left(k^2 - v^2\right)^{q-3} v^4}{9\gamma\alpha q\left(2k^2 - v^2\right)}\left(\frac{3\gamma}{2}\right)^q,$$

$$\left\{8\sqrt{2}v^4\left(\sqrt{2}v \mp 1\right)^2 (3\gamma/2)^q\left(1 \pm \sqrt{2}v\right)^{q-3}\right\}/$$
$$\left\{9\gamma q\alpha\left\{2\sqrt{2}\left[\pm(q-1)\gamma \mp 2\right]v^4 + 2\left[\gamma\left(2k^2 - 1\right)(q-1) + 4\left(1 - k^2\right)\right]v^3\right.\right.$$
$$+\sqrt{2}\left[\mp\gamma\left(1 + 2k^2\right)(q-1) \pm 2\left(4k^2 - 1\right)\right]v^2$$
$$+\left[\gamma\left(1 - 2k^2\right)(q-1) - 4k^2\right]v \pm \sqrt{2}(q-1)k^2\gamma\right\}\right\} \tag{19}$$

Inserting $H(t) = c_1 t^{2/(q-3)} + c_2 t^{n_2}$, with c_1 given by (17), into (15) yields $e_2 = (13 - 3q)/(q-3) + n_2$ for $n_2 \le (1+q)/(q-3)$, while $e_2 = 2(7-q)/(q-3)$ for $n_2 > (1+q)/(q-3)$. At the equality exponent $n_2 = (1+q)/(q-3)$, we obtain the subleading coefficient

$$c_2 = \frac{3\left(((q-1)\gamma - 2)v^2 + 4k^2\right)\gamma(q-3)}{4k^2(q-2)((q-1)\gamma - 4)}\left[\frac{9\gamma^3 v^4\left(9 - 6q + q^2\right)}{4k^2\alpha(\gamma - \gamma q + 2)}\right]^{\frac{2}{q-3}} \tag{20}$$

6 Conclusions

We have shown some problems that occur when dealing with generalized power expansions of solutions to nonlinear ordinary differential equations that depend on parameters, and we have sketched an algorithm that allows identifying critical values of the parameters and obtaining the series for the distinct families of solutions by an iterative process.

As an example, we have applied this algorithm to obtain two term truncations of the series expansions of solutions to a cosmological model filled with a nonlinear causal viscous fluid. Thus, it is shown as feasible to deal with the

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case branching that occurs in series solutions to nonlinear ordinary differential equations relevant to physical applications.

It deserves to be investigated how the complexity of the algorithm increases with the order of iteration, and whether this growth puts an effective limit to practical calculations. In such a case, it would be interesting to know whether more efficient algorithms could be devised. Also, it would be interesting to know whether expansions of solutions as shown in this paper can give information about the integrability of the equation.

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References

[1] D. Y. Grigoriev, and M. Singer Solving ordinary differential equations in terms of series with real exponents, Transactions of the A.M.S., 327(1), 1991, 329-351

[2] J. van der Hoeven, thesis Automatic asymptotics 1997, École polytechnique, France

[3] Loeb, Daniel E. Series with general exponents. Journal of Mathematical Analysis and Applications 156 (1991), no. 1, 184–208.

[4] A. S. Jakubi, Computer Physics Communications 115 (1998) 284.

[5] P. C. W. Davies *Phys. Lett* **68B** (1977) 402.

[6] L. P. Chimento *Class. Quantum Grav.* **6** (1989) 1285.

[7] D. Pavón, J. Bafaluy and D. Jou, *Class. Quantum Grav.* 8 (1991) 347.

[8] J. M. Aguirregabiria, A. Feinstein and J. Ibañez *Phys. Rev.* **D48** (1993) 4662.

[9] L. P. Chimento and A. S. Jakubi, *Class. Quantum Grav.* **10** (1993) 2047.

[10] M. Zakari and D. Jou, *Phys. Lett. A* **175** (1993) 395.

[11] M. Reuter and C. Wetterich *Phys. Lett. B* **188** (1987) 38.

[12] E. L. Ince, *Ordinary Differential Equations* (Dover, New York, 1956)

[13] W. Zimdahl, D. Pavón and R. Maartens, *Phys. Rev.* D **55** (1997) 4681.

[14] L. P. Chimento and A. S. Jakubi *Int. J. Mod. Phys.* D **8** (1999) 307.

[15] L. P. Chimento and A. S. Jakubi, *Proceedings of the First Mexican School on Gravitation and Mathematical Physics* (Guanajuato, Mexico, 1994) (World Scientific, 316 (1996).

[16] L.P. Chimento *J. Math. Phys.* 38 (1997) 2565.

[17] H. Yoshida, *Celest. Mech.* **31** (1983) 363.

[18] R. Maartens and V. Méndez *Phys. Rev.* D **55** (1997) 1937.

[19] R. Maartens *Class. Quantum Grav.* **12** (1995) 1455.

[20] L. P. Chimento, A. S. Jakubi, V. Mendez and R. Maartens *Class. Quantum Grav.* **14** (1997) 3363.